

A NEW WEIGHTED OSTROWSKI TYPE INEQUALITY INVOLVING INTEGRAL MEANS AND INTERVALS

A. QAYYUM^{1,2}, S. S. DRAGOMIR^{1,2}, M. SHOAIB, AND M. A. LATIF

ABSTRACT. The ostrowski inequality expresses bounds on the deviation of a function from its integral mean. The aim of this paper is to establish a new inequality using weight function which generalizes the inequalities of Dragomir, Wang and Cerone .The current article obtains bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval. A variety of earlier results are recaptured as particular instances of the current development. Applications for cumulative distribution function are also discussed.

1. INTRODUCTION

Since A. Ostrowski [14] proved his famous inequality in 1938, many mathematicians have been working about and around it, in many different directions and with applications in Numerical analysis and Probability etc. Several generalizations of the ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings and n-times differentiable mappings with error estimates for some special means and for some quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski inequality see[2]-[4] and [11]

The first (direct) generalization of Ostrowski's inequality was given by Milovanović and Pecarić in [12]. It was for the first time that Ostrowski-Grüss type inequality was given by Dragomir and Wang [7]. Cheng gave a sharp version of the mentioned inequality in [3]. Dragomir and Wang [5]-[8] and Cerone [1] pointed out a result to the above . Inspired and motivated by the work of Dragomir and Wang [5]-[8] and Cerone [1], we establish new inequalities, which are more generalized as compared to previous inequalities developed and discussed in [5]-[8]. Moreover, our results are in weighted form instead of previous results which are in non-weighted form.

The approach of Dragomir and Wang [5]-[8] and Cerone [1] for obtaining the bounds of a particular quadrature rule depended on the peano kernel while we use weighted peano kernel (see for example [10] and [15]) in our findings. This approach not only generalizes the results of Dragomir and Wang [5]-[8] and Cerone [1], but also gives some other interesting inequalities as special cases. Some closely related new results are also discussed. At the end, we will apply our main result for cumulative distribution function.

Ostrowski proved the following interesting and useful integral inequality:

Date: Today.

2000 Mathematics Subject Classification. Primary 65D30; Secondary 65D32.

Key words and phrases. ostrowski inequality, weight function, weighted integral mean.

This paper is in final form and no version of it will be submitted for publication elsewhere.

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) , whose derivative $f' : (a, b) \rightarrow \mathbb{R}$ is bounded on (a, b) , i.e. $\|f'\|_\infty = \sup_{t \in [a, b]} |f'(t)| < \infty$ then

$$(1.1) \quad |S(f; a, b)| \leq \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \frac{M}{b-a}$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one. Where the functional $S(f; a, b)$ represents the deviation of $f(x)$ from its integral mean over $[a, b]$ and be defined by

$$(1.2) \quad S(f; a, b) = f(x) - M(f; a, b),$$

and

$$(1.3) \quad M(f; a, b) = \frac{1}{b-a} \int_a^b f(x) dx.$$

In a series of papers, Dragomir and Wang [5]-[8] proved (1.1) and other variants for $f' \in L_p[a, b]$ for $p \geq 1$, the Lebesgue norms making use of a peano kernel approach and Montgomery's identity [15]. Montgomery's identity states that for absolutely continuous mappings $f : [a, b] \rightarrow \mathbb{R}$

$$(1.4) \quad f(x) = \frac{1}{b-a} \int_a^b f(t) dt + \frac{1}{b-a} \int_a^b P(x, t) f'(t) dt,$$

where the kernel $p : [a, b]^2 \rightarrow \mathbb{R}$ is given by

$$P(x, t) = \begin{cases} t-a & \text{if } a \leq t \leq x \leq b \\ t-b & \text{if } a \leq x < t \leq b \end{cases}$$

If we assume that $f' \in L_\infty[a, b]$ and $\|f'\|_\infty = \operatorname{ess} \sup_{t \in [a, b]} |f'(t)|$ then M in (1.1) may be replaced by $\|f'\|_\infty$.

Dragomir and Wang [5]-[8] utilizing an integration by parts argument, ostensibly Montgomery's identity (1.4), obtained

$$(1.1) \quad |S(f; a, b)|$$

$$\leq \begin{cases} \frac{1}{b-a} \left[\left(\frac{b-a}{2} \right)^2 + \left(x - \frac{a+b}{2} \right)^2 \right] \|f'\|_\infty, & f' \in L_\infty[a, b] \\ \frac{1}{b-a} \left[\frac{(x-a)^{q+1} + (b-x)^{q+1}}{q+1} \right]^{\frac{1}{q}} \|f'\|_p, & f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{b-a} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] \|f'\|_1, & \end{cases}$$

where $f : [a, b] \rightarrow \mathbb{R}$ is absolutely continuous on $[a, b]$ and the constants $\frac{1}{4}$, $\left[\frac{1}{q+1} \right]^{\frac{1}{q}}$ and $\frac{1}{2}$ respectively sharp.

Cerone [1], proved the following inequality:

Theorem 2. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous mapping and define

$$(1.6) \quad \tau(x; \alpha, \beta) := f(x) - \frac{1}{\alpha + \beta} [\alpha M(f; a, x) + \beta M(f; x, b)]$$

where

$$M(f; a, b) = \frac{1}{b-a} \int_a^b f(x) dx$$

then

$$(1.2) | \mathcal{D}(x; \alpha, \beta) | \leq \begin{cases} \frac{1}{2(\alpha+\beta)} [\alpha(x-a) + \beta(b-x)] \|f'\|_\infty, f' \in L_\infty[a, b] \\ \frac{1}{(\alpha+\beta)(q+1)^{\frac{1}{q}}} [\alpha^q(x-a) + \beta^q(b-x)]^{\frac{1}{q}} \|f'\|_p, f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} \left(1 + \frac{|\alpha-\beta|}{\alpha+\beta}\right) \|f'\|_1 \end{cases}$$

where $\|h\|$ are the usual Lebesgue norms for $h \in L[a, b]$ with

$$\|h\|_\infty := \operatorname{ess\,sup}_{t \in [a, b]} |h(t)| < \infty$$

and

$$\|h\|_p := \left(\int_a^b |h(t)|^p dt \right)^{\frac{1}{p}}, 1 \leq p \leq \infty$$

The current paper obtains bounds on the deviation of a function from weighted integral means from the end of the interval that cover the whole interval. The Ostrowski type results are recaptured as special cases.

2. MAIN RESULTS

To establish our main results we first give the following essential definitins and lemmas.

Definition 1. We assume that the weight function (or density) $w : (a, b) \rightarrow [0, \infty)$ to be non-negative and integrable over its entire domain and consider

$$\int_a^b w(t) dt < \infty.$$

The domain of w may be finite or infinite and may vanish at the boundary point. We denote the moments

$$\begin{aligned} m(a, b) &= \int_a^b w(t) dt, \\ N(a, b) &= \int_a^b f(t) w(t) dt \end{aligned}$$

Let the functional $S(f, w; a, b)$ be defined by

$$(2.1) \quad S(f, w; a, b) = f(x) - M(f, w; a, b)$$

where

$$(2.2) \quad M(f, w; a, b) = \frac{1}{\int_a^b w(x) dx} \int_a^b f(x) w(x) dx$$

The function $S(f, w; a, b)$ represents the deviation of $f(x)$ from its weighted integral mean over $[a, b]$.

We start with the following weighted identity which will be used to obtain bounds.

Lemma 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be an absolutely continuous mapping. Denoted by $P(x, \cdot) : [a, b] \rightarrow \mathbb{R}$ the weighted peano kernel is given by

$$(2.3) \quad \rho(x, t) = \begin{cases} \frac{\alpha}{\alpha + \beta} \frac{1}{m(a, x)} m(a, t), & a \leq t \leq x \\ \frac{\beta}{\alpha + \beta} \frac{1}{m(x, b)} m(b, t), & x < t \leq b \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$ are non negative and not both zero, then the weighted identity

$$(2.4) \quad \int_a^b P(x, t) f'(t) dt = f(x) - \frac{1}{\alpha + \beta} \left[\alpha \frac{N(a, x)}{m(a, x)} + \beta \frac{N(x, b)}{m(x, b)} \right]$$

holds.

Proof. From (2.3), we have

$$\begin{aligned} \int_a^b P(x, t) f'(t) dt &= \frac{\alpha}{\alpha + \beta} \frac{1}{m(a, x)} \int_a^x \left(\int_a^t w(u) du \right) f'(t) dt \\ &\quad + \frac{\beta}{\alpha + \beta} \frac{1}{m(x, b)} \int_x^b \left(\int_b^t w(u) du \right) f'(t) dt \end{aligned}$$

where the integration by parts formula has been utilized on the separate intervals $[a, x]$ and $(x, b]$. Simplification of the expressions readily produces the identity as stated. \square

We now give our main result.

Theorem 3. Let $f : [a, b] \rightarrow \mathbb{R}$ be absolutely continuous mapping and define

$$(2.5) \quad \tau(x, w; \alpha, \beta) := f(x) - \frac{1}{\alpha + \beta} [\alpha M(f, w; a, x) + \beta M(f, w; x, b)]$$

where $M(f, w; a, b)$ is the weighted integral mean defined in (2.2), then

$$(2.1) \quad |\tau(x, w; \alpha, \beta)| \leq \begin{cases} \frac{1}{2(\alpha+\beta)} \left[\frac{\alpha}{m(a, x)} (x-a)^2 + \frac{\beta}{m(x, b)} (b-x)^2 \right] w(x) \|f'\|_\infty \\ \frac{1}{(q+1)^{\frac{1}{q}} (\alpha+\beta)} \left[\left(\frac{\alpha^q}{m(a, x)} (x-a)^2 + \frac{\beta^q}{m(x, b)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}} \|f'\|_p \\ \frac{1}{2} \left(1 + \frac{|\alpha-\beta|}{\alpha+\beta} \right) \|f'\|_1 \end{cases}$$

Proof. Taking the modulus of (2.4), we have from (2.5) and (2.2)

$$(2.7) \quad |\tau(x, w; \alpha, \beta)| = \left| \int_a^b P(x, t) f'(t) dt \right| \leq \int_a^b |P(x, t)| |f'(t)| dt,$$

where we have used the well known properties of the integral and modulus.

Thus, for $f' \in L_\infty[a, b]$ from (2.7) gives

$$|\tau(x, w; \alpha, \beta)| \leq \|f'\|_\infty \int_a^b |P(x, t)| dt,$$

from which a simple calculation using (2.3) gives

$$\begin{aligned} & \int_a^b |P(x, t)| dt \\ &= \frac{\alpha}{\alpha+\beta} \frac{1}{m(a, x)} \int_a^x \left(\int_a^t w(u) du \right) dt + \frac{\beta}{\alpha+\beta} \frac{1}{m(x, b)} \int_x^b \left(\int_t^b w(u) du \right) dt \\ &= \left[\frac{\alpha}{m(a, x)} (x-a)^2 + \frac{\beta}{m(x, b)} (b-x)^2 \right] \frac{w(x)}{2(\alpha+\beta)}. \end{aligned}$$

Hence the first inequality is obtained.

$$|\tau(x, w; \alpha, \beta)| \leq \frac{1}{2(\alpha+\beta)} \left[\frac{\alpha}{m(a, x)} (x-a)^2 + \frac{\beta}{m(x, b)} (b-x)^2 \right] w(x) \|f'\|_\infty$$

Further, using Hölder's integral inequality, from (2.7) we have for $f' \in L_p[a, b]$

$$|\tau(x, w; \alpha, \beta)| \leq \|f'\|_p \left(\int_a^b |P(x, t)|^q dt \right)^{\frac{1}{q}}.$$

where $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Now

$$\begin{aligned}
& (\alpha + \beta) \left(\int_a^b |P(x, t)|^q dt \right)^{\frac{1}{q}} \\
&= \left[\alpha^q \frac{1}{m(a, x)} \int_a^x \left(\int_a^t w(u) du \right)^q dt + \beta^q \frac{1}{m(x, b)} \int_x^b \left(\int_t^b w(u) du \right)^q dt \right]^{\frac{1}{q}} \\
&= \frac{1}{(q+1)^{\frac{1}{q}}} \left[\left(\frac{\alpha^q}{m(a, x)} (x-a)^2 + \frac{\beta^q}{m(x, b)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}}
\end{aligned}$$

and so the second inequality is obtained

$$\begin{aligned}
& |\tau(x; \alpha, \beta)| \\
&\leq \frac{1}{(q+1)^{\frac{1}{q}} (\alpha + \beta)} \left[\left(\frac{\alpha^q}{m(a, x)} (x-a)^2 + \frac{\beta^q}{m(x, b)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}} \|f'\|_p
\end{aligned}$$

Finally, for $f' \in L_1[a, b]$, we have from (2.7) and using (2.3)

$$|\tau(x, w; \alpha, \beta)| \leq \sup_{t \in [a, b]} |P(x, t)| \|f'\|_1$$

where

$$\begin{aligned}
(\alpha + \beta) \sup_{t \in [a, b]} |P(x, t)| &= \max \{ \alpha, \beta \} = \frac{\alpha + \beta}{2} + \left| \frac{\alpha - \beta}{2} \right| \\
&= \left(1 + \frac{|\alpha - \beta|}{\alpha + \beta} \right) \frac{\|f'\|_1}{2}
\end{aligned}$$

This completes the proof of theorem. \square

Remark 1. If we put $w(x) = 1$, in (2.6), we get cerone's result (1.7). If we put $\alpha = \beta$ and $w(x) = 1$, in (2.6), we get Dragomir's result (1.5). Similarly, for different weights, we can obtain a variety of results.

Remark 2. It should be noted that from (2.5) and (2.1)

$$(2.8) \quad (\alpha + \beta) \tau(x, w; \alpha, \beta) = \alpha S(f, w; a, x) + \beta S(f, w; x, b)$$

From (1.5) using the triangle inequality, we obtain

$$(2.2) \quad |(\alpha + \beta) \tau(x, w; \alpha, \beta)|$$

$$\leq \begin{cases} \frac{\alpha}{2} \frac{1}{m(a, x)} (x-a)^2 w(x) \|f'\|_{\infty, [a, x]} + \frac{\beta}{2} \frac{1}{m(x, b)} (b-x)^2 w(x) \|f'\|_{\infty, [x, b]} \\ \alpha \left[\left(\frac{1}{m(a, x)(q+1)} (x-a)^2 \right) w(x) \right]^{\frac{1}{q}} \|f'\|_{p, [a, x]} \\ + \beta \left[\left(\frac{1}{m(x, b)(q+1)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}} \|f'\|_{p, [x, b]} \\ \alpha \|f'\|_{1, [a, x]} + \beta \|f'\|_{1, [x, b]} \end{cases}$$

where for $[c, d] \subseteq [a, b]$

$$\|h\|_{p,[c,d]} := \left(\int_c^d |h(t)|^p dt \right)^{\frac{1}{p}}, p \geq 1$$

and $\|h\|_{\infty,[c,d]} := \sup_{t \in [c,d]} |h(t)|$

That is,

$$(2.3) \quad |(\alpha + \beta) \tau(x, w; \alpha, \beta)| \leq \begin{cases} \left(\frac{\alpha}{m(a,x)} (x-a)^2 + \frac{\beta}{m(x,b)} (b-x)^2 \right) w(x) \frac{\|f'\|_{\infty}}{2} \\ \alpha \left[\left(\frac{1}{m(a,x)(q+1)} (x-a)^2 \right) w(x) \right]^{\frac{1}{q}} + \beta \left[\left(\frac{1}{m(x,b)(q+1)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}} \|f'\|_p \\ (\alpha + \beta) \|f'\|_1 \end{cases}$$

where the expression (2.10) involving the $\|\cdot\|_p$ norm is coarser.

The results of (2.9) in which the norms are evaluated over the two subintervals, although finer, they do require more work.

Remark 3. It is possible to reduce the amount of work alluded to in Remark 2, since we may write

$$\begin{aligned} & \alpha M(f, w; a, x) + \beta M(f, w; x, b) \\ = & \alpha M(f, w; a, x) + \frac{\beta}{m(x, b)} \left[\int_a^b f(u) w(u) du - \int_a^x f(u) w(u) du \right] \\ = & \alpha M(f, w; a, x) - \frac{\beta}{m(x, b)} \int_a^x f(u) w(u) du + \frac{\beta}{m(x, b)} \int_a^b f(u) w(u) du \\ = & (\alpha + \beta - \beta \sigma_w(x)) M(f, w; a, x) + \beta \sigma_w(x) M(f, w; a, b) \end{aligned}$$

where

$$(2.11) \quad \frac{m(a, b)}{m(x, b)} = \sigma_w(x)$$

Thus, from (2.5),

$$(2.4) \quad \begin{aligned} & \tau(x, w; \alpha, \beta) \\ = & \frac{1}{\alpha + \beta} \left(\frac{\alpha}{x-a} m(a, x) + \frac{\beta}{b-x} m(x, b) \right) f(x) \\ & - \left[\left(1 - \frac{\beta}{\alpha + \beta} \sigma_w(x) \right) M(f, w; a, x) + \frac{\beta}{\alpha + \beta} \sigma_w(x) M(f, w; a, b) \right] \end{aligned}$$

so that for fixed $[a, b]$, $M(f, w; a, b)$ is also fixed.

Corollary 1. *Let the condition of Theorem 3 holds. then*

$$(2.5) \quad \left| f(x) - \frac{1}{2} [M(f, w; a, x) + M(f, w; x, b)] \right| \leq \begin{cases} \left[\frac{1}{m(a, x)} (x-a)^2 + \frac{1}{m(x, b)} (b-x)^2 \right] \frac{w(x) \|f'\|_\infty}{4} \\ \left[\left(\frac{1}{m(a, x)} (x-a)^2 + \frac{1}{m(x, b)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}} \frac{\|f'\|_p}{2(q+1)^{\frac{1}{q}}} \\ \frac{\|f'\|_1}{2} \end{cases}$$

Proof. The result is readily obtained on allowing $\alpha = \beta$ in (2.6) so that the left hand side is $\tau(x; \alpha, \alpha)$ from (2.5). \square

Corollary 2. *Let the conditions of Theorem 3 hold. Then*

$$(2.6) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{\alpha+\beta} \left[\alpha M\left(f, w; a, \frac{a+b}{2}\right) + \beta M\left(f, w; \frac{a+b}{2}, b\right) \right] \right| \leq \begin{cases} \frac{1}{2(\alpha+\beta)} \left[\frac{\alpha}{m(a, \frac{a+b}{2})} \left(\frac{b-a}{2}\right)^2 + \frac{\beta}{m(\frac{a+b}{2}, b)} \left(\frac{b-a}{2}\right)^2 \right] w\left(\frac{a+b}{2}\right) \|f'\|_\infty, f' \in L_\infty[a, b] \\ \frac{1}{(q+1)^{\frac{1}{q}}(\alpha+\beta)} \left[\left(\frac{\alpha^q}{m(a, x)} \left(\frac{b-a}{2}\right)^2 + \frac{\beta^q}{m(x, b)} \left(\frac{b-a}{2}\right)^2 \right) w\left(\frac{a+b}{2}\right) \right]^{\frac{1}{q}} \|f'\|_p \\ , f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} \left(1 + \frac{|\alpha-\beta|}{\alpha+\beta} \right) \|f'\|_1 \end{cases}$$

Proof. Placing $x = \frac{a+b}{2}$ in (2.5) and (2.6) produces the results stated in (2.14). \square

Corollary 3. *If (2.13) is evaluated at the midpoint then*

$$(2.7) \quad \left| f\left(\frac{a+b}{2}\right) - \frac{1}{2} M(f, w; a, b) \right| \leq \begin{cases} \frac{1}{2(\alpha+\beta)} \left[\frac{\alpha}{m(a, \frac{a+b}{2})} \left(\frac{b-a}{2}\right)^2 + \frac{\beta}{m(\frac{a+b}{2}, b)} \left(\frac{b-a}{2}\right)^2 \right] w\left(\frac{a+b}{2}\right) \|f'\|_\infty \\ \frac{1}{(q+1)^{\frac{1}{q}}(\alpha+\beta)} \left[\left(\frac{\alpha^q}{m(a, x)} \left(\frac{b-a}{2}\right)^2 + \frac{\beta^q}{m(x, b)} \left(\frac{b-a}{2}\right)^2 \right) w\left(\frac{a+b}{2}\right) \right]^{\frac{1}{q}} \|f'\|_p \\ \frac{1}{2} \|f'\|_1 \end{cases}$$

which is in agreement with (1.5) when $x = \frac{a+b}{2}$. The above result can also be obtained by taking $\alpha = \beta$ in (2.14) or equivalently $\alpha = \beta$ and $x = \frac{a+b}{2}$ in (2.6).

3. An Application to the Weighted Cumulative Distribution Function

Let X be a random variable taking values in the finite interval $[a, b]$ with Cumulative Distributive Function

$$F_w(x) = P_r(X \leq x) = \int_a^x f(u) w(u) du,$$

we also use the fact that

$$\int_a^b f(u) w(u) du = 1$$

where f is a Probability Density Function. The following theorem holds.

Theorem 4. *Let X and F be as above, then*

$$(3.1) \quad |[\alpha m(x, b) - \beta m(a, x)] F_w(x) - m(a, x) [(\alpha + \beta) m(x, b) f(x) - \beta]|$$

$$\leq \begin{cases} \frac{1}{2} [\alpha m(x, b) (x - a)^2 + \beta m(a, x) (b - x)^2] w(x) \|f'\|_\infty, f' \in L_\infty[a, b] \\ \frac{1}{(q+1)^{\frac{1}{q}}} m(x, b) m(a, x) \left[\left(\frac{\alpha^q}{m(a, x)} (x - a)^2 + \frac{\beta^q}{m(x, b)} (b - x)^2 \right) w(x) \right]^{\frac{1}{q}} \|f'\|_p \\ , f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} m(x, b) m(a, x) (\alpha + \beta + |\alpha - \beta|) \|f'\|_1, f' \in L_1[a, b] \end{cases}$$

Proof. From (2.5), we have

$$\begin{aligned} & \tau(x, w; \alpha, \beta) \\ & : = f(x) - \frac{1}{\alpha + \beta} [\alpha M(f, w; a, x) + \beta M(f, w; x, b)] \\ & = f(x) - \frac{1}{\alpha + \beta} \left[\frac{\alpha}{m(a, x)} \int_a^x f(u) w(u) du + \frac{\beta}{m(x, b)} \int_x^b f(u) w(u) du \right]. \end{aligned}$$

Or

$$\begin{aligned} & -(\alpha + \beta) m(a, x) m(x, b) \tau(x, w; \alpha, \beta) \\ & = (\alpha m(x, b) - \beta m(a, x)) F_w(x) - (\alpha + \beta) m(a, x) m(x, b) f(x) + \beta m(a, x) \\ & = (\alpha m(x, b) - \beta m(a, x)) F_w(x) - m(a, x) [(\alpha + \beta) m(x, b) f(x) - \beta]. \end{aligned}$$

The proof follows in a straightforward manner from (2.6).

Using (2.12) for $\tau(x, w; \alpha, \beta)$ and (2.13). Taking the modulus and using (2.6) gives the stated result. \square

Corollary 4. *Let X be a random variable, $F(w, x)$ weighted Cumulative Distributive Function and f is a Probability Density Function. Then*

$$(3.2) \quad \left| \frac{1}{2} [m(x, b) - m(a, x)] F_w(x) - m(a, x) \left[m(x, b) f(x) - \frac{1}{2} \right] \right|$$

$$\leq \begin{cases} \left[\frac{1}{4} m(x, b) (x-a)^2 + m(a, x) (b-x)^2 \right] w(x) \|f'\|_\infty, f' \in L_\infty[a, b] \\ m(x, b) m(a, x) \frac{1}{2(q+1)^{\frac{1}{q}}} \left[\left(\frac{1}{m(a, x)} (x-a)^2 + \frac{1}{m(x, b)} (b-x)^2 \right) w(x) \right]^{\frac{1}{q}} \\ \times \|f'\|_p, f' \in L_p[a, b], p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \frac{1}{2} m(x, b) m(a, x) \|f'\|_1, f' \in L_1[a, b] \end{cases}$$

Remark 4. *The above result allow the approximation of $F(x)$ in terms of $f(x)$. The approximation of*

$$R_w(x) = 1 - F_w(x)$$

could also be obtained by a simple substitution. $R_w(x)$ is of importance in reliability theory where $f(x)$ is the p.d.f of failure.

Remark 5. *We may directly put $\beta = 0$ in (2.5) and (2.6), assuming that $\alpha \neq 0$ to obtain*

$$(3.3) \quad \left(\frac{1}{x-a} m(a, x) \right) f(x) - F_w(x)$$

$$\leq \begin{cases} \frac{(x-a)^2}{2} w(x) \|f'\|_\infty \\ (x-a)^{1+\frac{1}{q}} w(x) \|f'\|_p \frac{1}{(q+1)^{\frac{1}{q}}} \\ (x-a) \|f'\|_1 \end{cases}$$

which agrees with (1.5) for $|S(f; a, x)|$.

We may replace f by F in any of the equations (3.1) – (3.3) so that the bounds are in terms of $\|f'\|_p, p \geq 1$. Further we note that

$$\int_a^b F_w(u) du = u F_w(u) \Big|_a^b - \int_a^b x w(x) f(x) dx = b - E[Xw(X)].$$

Conclusion: Cerone [1], obtained bounds for the deviation of a function from a combination of integral means over the end intervals covering the entire interval and applied these results to approximate the cumulative distribution function in terms of the probability density function. Inspired and motivated by the work of Cerone [1], we establish a new inequality, which is more generalized as compared to previous inequalities developed in [5]-[8]. In addition, the approach of Cerone [1] for obtaining the bounds of a particular quadrature rule has depended on the peano kernel but we use weighted peano kernel in our findings. This approach not only generalizes the results of [1] but also gives some other interesting inequalities as special cases. Approximation of the weighted cumulative distribution functions in terms of weighted probability density function is given as well.

REFERENCES

- [1] P. Cerone, A new Ostrowski Type Inequality Involving Integral Means Over End Intervals, Tamkang Journal Of Mathematics Volume 33, Number 2, 2002.
- [2] P. Cerone and S.S. Dragomir, Trapezoidal type rules from an inequalities point of view, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press N.Y. (2000).
- [3] X.L. Cheng, Improvement of some Ostrowski-Grüss type inequalities, Comput. Math. Appl. 42 (2001), 109-114.
- [4] S.S. Dragomir and N. S. Barnett, An Ostrowski type inequality for mappings whose second derivatives are bounded and applications, RGMIA Research Report Collection, V.U.T., 1(1999), 67-76.
- [5] S. S. Dragomir and S. Wang, A new inequality Ostrowski's type in L_p norm, Indian J. of Math. 40(1998), 299-304.
- [6] S. S. Dragomir and S. Wang, A new inequality Ostrowski's type in L_1 norm and applications to some special means and some numerical quadrature rules, Tamkang J. of Math. 28(1997), 239-244.
- [7] S. S. Dragomir and S. Wang, An inequality Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Computers Math. Applic. 33(1997), 15-22.
- [8] S. S. Dragomir and S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and to some numerical quadrature rules, Appl. Math. Lett. 11(1998), 105-109.
- [9] S.S. Dragomir and S. Wang, An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules, Comput. Math. Appl., 33 (11), 15-20, (1997).
- [10] S. Hussain and A. Qayyum: A generalized Ostrowski-Grüss type inequality for bounded differentiable mappings and its applications. Journal of Inequalities and Applications 2013:1.
- [11] Z. Liu, Some companions of an Ostrowski type inequality and application, J. Inequal. in Pure and Appl. Math, 10(2), 2009, Art. 52, 12 pp.
- [12] G.V. Milovanović and J. E. Pečarić, On generalization of the inequality of A. Ostrowski and some related applications, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. Fiz. (544-576), 155-158, (1976).
- [13] D. S. Mitrinović, J. E. Pečarić and A. M. Fink, Inequalities for Functions and Their Integrals and Derivatives, Kluwer Academic Publishers, 1994.
- [14] A. Ostrowski, Über die Absolutabweichung einer differentiellen Funktionen von ihren Integralmitteln, Comment. Math. Hel. 10(1938), 226-227.
- [15] Ather Qayyum and Sabir Hussain, A new generalized Ostrowski Grüss type inequality and applications, Applied Mathematics Letters 25 (2012) 1875-1880.

¹DEPARTMENT OF MATHEMATICS, UNIVERSITI TEKNOLOGY PATRONAS, MALAYSIA. ²DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIL, HAIL 2440, SAUDI ARABIA

E-mail address: atherqayyum@gmail.com

¹MATHEMATICS, COLLEGE OF ENGINEERING & SCIENCE, VICTORIA UNIVERSITY, PO Box 14428, MELBOURNE CITY, MC 8001, AUSTRALIA., ²SCHOOL OF COMPUTATIONAL AND APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, JOHANNESBURG 2050, SOUTH AFRICA.

E-mail address: sever.dragomir@vu.edu.au

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIL, HAIL 2440, SAUDI ARABIA

E-mail address: safridi@gmail.com

SCHOOL OF COMPUTATIONAL AND APPLIED MATHEMATICS, UNIVERSITY OF THE WITWATERSRAND, PRIVATE BAG 3, WITS 2050, JOHANNESBURG, SOUTH AFRICA

E-mail address: m_amer_latif@hotmail.com